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Wiener Sausage Volume Moments

A. M. Berezhkovskii,[1] Yu. A. Makhnovskii,[1] and R. A. Suris[1]

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The statistical characteristics of a spatial region visited by a spherical Brownian particle during time t (Wiener sausage) are investigated. The expectation value and dispersion of this quantity are obtained for a space of arbitrary dimension. In the one-dimensional case the distribution of probability density and the moments of any order are determined for this quantity.

KEY WORDS: Brownian movement of spherical particle; Wiener sausage.

1. INTRODUCTION

In the theory of random processes the spatial region visited by a spherical Brownian particle during time t is known as the Wiener sausage.[1,2] Its volume is important in an analysis of a number of physical processes. The average value of this random quantity was calculated for the first time in a pioneering work[3] for the two-dimensional case. In the present study we investigate the statistical characteristics of the Wiener sausage volume in a space of arbitrary dimension. The results are described in the following order. Sections 2 and 3 deal with the determination of the expectation value and dispersion of the quantity under consideration at asymptotically large times. A one-dimensional case is analyzed in Section 4, where the distribution probability density and the moments of any order are calculated. The concluding Section 5 contains a discussion of the relationship between the volume of the Wiener sausage and the number of different sites visited by a random walk on a lattice.

2. AVERAGE VOLUME OF A WIENER SAUSAGE

Let us consider a spherical Brownian particle with a radius b. We introduce $\varphi(\mathbf{r}, W_t)$, which is a function of the position r of the point in

[1] Karpov Institute of Physical Chemistry, 103064 Moscow, USSR.

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d-dimensional space and a functional of the Wiener trajectory W_t of the particle center observed during time t:

$$\varphi(\mathbf{r}, W_t) = \begin{cases} 1 & \text{if} \quad \min |\mathbf{r} - \mathbf{r}_{W_t}| \leqslant b \\ 0 & \text{if} \quad \min |\mathbf{r} - \mathbf{r}_{W_t}| > b \end{cases} \tag{1}$$

where $\mathbf{r}_{W_t} \in W_t$. This makes it possible to determine the volume of the Wiener sausage that corresponds to a given trajectory W_t in the form

$$v(W_t) = \int \varphi(r, W_t)\, d^d r \tag{2}$$

The random value $v(W_t)$ is distributed with the probability density

$$F_t(v) = \langle \delta(v - v(W_t)) \rangle \tag{3}$$

The symbol $\langle \cdots \rangle$ stands for the average with respect to the Wiener trajectories.[4,5] The function $F_t(v)$ is normalized with respect to unity; and at the starting moment $t = 0$ it is $\delta(v - v_0)$, where $v_0 = \pi^{d/2} b^d / \Gamma(1 + d/2)$ is the volume of the Brownian particle.

In accordance with definitions (1)–(3), the average volume of the Wiener sausage is

$$\bar{v}(t) = \int v F_t(v)\, dv = \langle v(W_t) \rangle = \int \langle \varphi(\mathbf{r}, W_t) \rangle\, d^d r \tag{4}$$

The quantity $\langle \varphi(\mathbf{r}; W_t) \rangle$ is the portion of the trajectories which during time t visited the b neighborhood of the point $\mathbf{r}$ at least once. This quantity equals the probability of the death during time t of a Brownian point particle in the sink of radius b around the $\mathbf{r}$ point. The survival probability of such a particle is an integral over the whole space of the probability density of finding the particle at the point $\mathbf{r}'$ at moment t—the Green function $G(\mathbf{r}', t)$.[2] This function obeys the diffusion equation

$$\partial G / \partial t = D\, \Delta G \tag{5}$$

(where D is the diffusion coefficient), the initial condition

$$G(\mathbf{r}', 0) = \delta(\mathbf{r}) \tag{6}$$

and the boundary conditions

$$G(|\mathbf{r} - \mathbf{r}'| = b, t) = 0, \ G(\infty, t) = 0$$

[2] Without loss of generality, we assume that the particles start from the origin of the coordinates at the moment $t = 0$.

It is convenient to express the results of the calculation of $\langle \varphi(\mathbf{r}; W_t) \rangle$ in the following form:

$$\langle \varphi(\mathbf{r}; W_t) \rangle = \theta(r-b)\, u(r, t) + \theta(b-r) \tag{7}$$

Through the Heaviside step function $\theta(x)$, Eq. (7) reflects the fact that the falling of the starting point of the particle into the sink results in its instant death. Here we have

$$u(r, \tau) = \frac{2}{\pi} \left(\frac{b}{r}\right)^{\nu} \int_0^{\infty} \frac{1 - \exp(-\tau y)}{y} \times \frac{J_\nu(y)\, N_\nu((r/b)\, y) - J_\nu((r/b)\, y)\, N_\nu(y)}{J_\nu^2(y) + N_\nu^2(y)}\, dy \tag{8}$$

where $\nu = (d-2)/2$, $\tau = Dt/b^2$, and J_ν and N_ν are Bessel functions of the first and second kinds of order of ν. By substituting Eq. (7) into Eq. (4) and integrating,[3] we obtain

$$\bar{v}(\tau) = v_0 \left\{ 1 + d \left[\tau(d-2)\, \theta(d-2) + \frac{4}{\pi^2} \int_0^{\infty} \frac{1 - \exp(-\tau x^2)}{J_\nu^2(x) + N_\nu^2(x)} \frac{dx}{x^3} \right] \right\} \tag{9}$$

This equation determines the average volume of the region in d-dimensional space visited by the Brownian particle during time t. When $d = 1, 3$, Eq. (9) acquires an especially simple form:

$$\bar{v}(\tau) = v_0 \left(1 + \frac{2}{\sqrt{\pi}} \sqrt{\tau} \right) \qquad \text{when} \quad d = 1 \tag{10}$$

$$\bar{v}(\tau) = v_0 \left(1 + \frac{6}{\sqrt{\pi}} \sqrt{\tau} + 3\tau \right) \qquad \text{when} \quad d = 3 \tag{11}$$

Next we analyze the $\bar{v}(\tau)$ dependence. At small values of time

$$\tau \ll 1/(d-1)^2 \qquad \text{when} \quad d \geqslant 2 \tag{12}$$

the asymptotic $\bar{v}(\tau)$ is universal:

$$\bar{v}(\tau) = v_0 \left[1 + \frac{2d}{\sqrt{\pi}} \sqrt{\tau} + O(\tau) \right]$$

This reflects the one-dimensional nature of the Brownian motion at these moments in a space of an arbitrary dimension. According to Eq. (12), the

[3] It is convenient first to integrate over the volume of the Laplace transform [Eq. (7)] and then to carry out the reverse Laplace transformation.

time interval where this holds true is the smaller, the greater the dimension of the space.

Of special interest is the behavior of $\bar{v}(\tau)$ at large time values, $\tau \gg 1$. In a two-dimensional case the quantity under consideration can be represented in the form of an asymptotic series (we perform an inverse Laplace transformation by a special method[6])

$$\bar{v}(\tau) \simeq v_0 \frac{4\tau}{\ln \beta\tau} \sum_{j=0}^{N} \frac{A_j}{\ln^j \beta\tau} \tag{13}$$

where $\beta = 4\exp(-2C) \simeq 1.261$; $C \simeq 0.577$ is the Euler constant; and the coefficient A_j are described by the following equation:

$$A_j = \frac{d^j}{dx^j}\left[\frac{1}{\Gamma(1-x)}\right]_{x=-1}$$

In particular, $A_0 = 1$, $A_1 \simeq 0.423$, $A_2 \simeq 0.466$, $A_3 \simeq 1.147$, and $A_4 \simeq -0.589$. For $N = 2$, Eq. (13) coincides with the known result.[3] For $d \geqslant 3$ the main term of the asymptotics of for large time values becomes universal:

$$\bar{v}(\tau) \simeq v_0\, d(d-2)\tau \tag{14}$$

The calculation of the Wiener sausage volume dispersion requires that account should be taken of the correction terms. From Eq. (9) it follows that

$$\bar{v}(\tau) \simeq v_0 \cdot 8\tau\left[1 + \frac{\ln \beta\tau}{\tau} + \frac{2C+1}{8\tau} + \frac{\ln \beta\tau}{8\tau^2} + O\left(\frac{1}{\tau^2}\right)\right], \qquad d = 4 \tag{15}$$

$$\bar{v}(\tau) \simeq v_0 d(d-2)\tau\left[1 + \frac{2}{d(d-4)}\frac{1}{\tau} + o\left(\frac{1}{\tau}\right)\right], \qquad d \geqslant 5 \tag{16}$$

Thus, the $\bar{v}(\tau)$ function for large time values in spaces of different dimensions has the following form. The main term of the asymptotics increases with time as $\tau^{1/2}$ in the one-dimensional case; as $\tau/\ln \tau$ in the two-dimensional case; and is proportional to τ for $d \geqslant 3$. The correction term for the linear dependence $\bar{v}(\tau)$ $(d \geqslant 3)$ increases as $\tau^{1/2}$ in three-dimensional space; it increases as $\ln \tau$ in the four-dimensional case; and for $d \geqslant 5$ is independent of time, decreasing with an increase in the dimension of space.

3. VOLUME DISPERSION OF A WIENER SAUSAGE

Next we calculate the second moment of the volume of a Wiener sausage, $v^2(t)$. According to the definitions in Eqs. (1)–(3),

$$\overline{v^2}(t) = \int v^2 F_t(v)\, dv = \langle v^2(W_t)\rangle$$

$$= \int \langle \varphi(\mathbf{r}_1, W_t)\, \varphi(\mathbf{r}_2, W_t)\rangle\, d^d r_1\, d^d r_2 \tag{17}$$

The quantity $\langle \varphi(\mathbf{r}_1, W_t)\, \varphi(\mathbf{r}_2, W_t)\rangle$ is the portion of the trajectory which during time t has visited at least once the b neighborhood of both the $\mathbf{r}_1$ and the $\mathbf{r}_2$ points. This quantity can be expressed through the death probability of a particle in a one-sink and a two-sink situation. To do this we express the quantity in the form

$$\langle \varphi(\mathbf{r}_1, W_t)\, \varphi(\mathbf{r}_2, W_t)\rangle = \langle \varphi(\mathbf{r}_1, W_t)\rangle + \langle \varphi(\mathbf{r}_2, W_t)\rangle$$
$$- \{1 - \langle [1 - \varphi(\mathbf{r}_1, W_t)][1 - \varphi(\mathbf{r}_2, W_t)]\rangle\} \tag{18}$$

The first two terms on the right-hand side of this equation are equal to the death probability of a Brownian point particle during time t on one sink of radius b located at $\mathbf{r}_1$ and $\mathbf{r}_2$ points, respectively. The quantity in braces is the death probability of the particle during the same time in a two-sink situation. Indeed, the average quantity $\langle [1 - \varphi(\mathbf{r}_1; W_t)][1 - \varphi(\mathbf{r}_2; W_t)]\rangle$ represents the portion of a trajectory which during time t does not visit the b neighborhood of either the $\mathbf{r}_1$ or the $\mathbf{r}_2$ point. This quantity is equal to the survival probability of the particle in the presence of two sinks located at the $\mathbf{r}_1$ and $\mathbf{r}_2$ points.

The death probability of a particle in a one-sink situation is known [cf. Eq. (7)]. In a two-sink situation it has been impossible to calculate the death probability for an arbitrary position of the sinks. However, we do not need the death probability itself, but an integral of it taken over different configurations of the sinks. At large time values ($\tau \gg 1$) the main contribution to the integral comes from the configuration in which all the characteristic lengths considerably exceed the dimension of the sink, i.e.,

$$r_1 \gg b; \qquad r_2 \gg b; \qquad R = |\mathbf{r}_1 - \mathbf{r}_2| \gg b \tag{19}$$

For these configurations an approximate calculation of the death probability of the particle is possible. This makes it possible to elucidate the behavior of the mean square volume of a Wiener sausage at large time values.

The calculations presented in the Appendix show that for a two-sink situation which satisfies the conditions (19), the Laplace transform of the particle's death probability during time t,

$$u(r_1, r_2, t) = 1 - \langle [1 - \varphi(r_1, W_t)][1 - \varphi(r_2, W_t)] \rangle$$

is approximately equal to

$$u(r_1, r_2, s) = \int_0^\infty u(r_1, r_2, t) \exp(-ts)\, dt \simeq \frac{u(r_1, s) + u(r_2, s)}{1 + su(R, s)} \tag{20}$$

Here, $u(r, s)$, the Laplace transform of the $u(r, t)$ function [Eq. (8)], is

$$u(r, s) = \frac{1}{s} \left(\frac{b}{r}\right)^\nu \frac{K_\nu((sr^2/D)^{1/2})}{K_\nu((sb^2/D)^{1/2})}$$

where $K_\nu(z)$ is the MacDonald function of order ν. We shall utilize Eq. (20) for any configuration of a pair of sinks, including those for which the conditions (19) are not fulfilled. A rough estimation shows that in so doing an error of the order of $v_0 \bar{v}(t)$ occurs in calculating $\overline{v^2}(t)$.

According to Eqs. (17), (18), and (20), the Laplace transform of the second moment of the volume of a Wiener sausage for $s \ll D/b^2$ has the following form:

$$v^2(s) \simeq \int \frac{su(|\mathbf{r}_1 - \mathbf{r}_2|, s)[u(r_1, s) + u(r_2, s)]}{1 + su(|\mathbf{r}_1 - \mathbf{r}_2|, s)}\, d^d r_1\, d^d r_2$$

Since with the accepted precision $\int u(r, s)\, d^d r \simeq \bar{v}(s)$, then

$$\overline{v^2} \simeq 2\bar{v}(s) \int \frac{su(R, s)}{1 + su(R, s)}\, d^d R \tag{21}$$

Equation (21) makes it possible to find the $\overline{v^2}(t)$ function at large t in spaces of different dimensions.

We shall use this possibility for calculating the volume dispersion of a Wiener sausage:

$$\sigma^2(t) = \overline{v^2}(t) - \bar{v}(t)^2$$

For the one-dimensional case it follows from Eqs. (10) and (21) that

$$\overline{v^2}(\tau) \simeq v_0^2[2\tau \ln 2 + O(\sqrt{\tau})] \tag{22}$$

$$\sigma^2(\tau) \simeq v_0^2 \left[2 \left(\ln 2 - \frac{2}{\pi} \right) \tau + O(\sqrt{\tau}) \right] \tag{23}$$

It should be noted that the main terms in these equations coincide with the result of a precise calculation carried out in the following section.

In the two-dimensional case, after performing an inverse Laplace transformation [Eq. (21)] by a special method,[6] we obtain the representation $\overline{v^2}(\tau)$ in the form of an asymptotic series:

$$\overline{v^2}(\tau) \simeq v_0^2 \frac{16\tau^2}{\ln^2 \beta\tau} \sum_{j=0}^{N} \frac{B_j}{\ln^j \beta\tau} \tag{24}$$

It is more cumbersome to calculate the coefficients of these series than to calculate the coefficients of an asymptotic series [Eq. (13)] for $\bar{v}(\tau)$. We have calculated the first four of them, which are necessary for obtaining the main terms in the dispersion:

$$B_0 = 1; \qquad B_1 = 2A_1 = 2(1 - C) \simeq 0.845$$

$$B_2 = 2\left[\frac{4}{3} I_3 + A_1^2 - \left(\frac{\pi^2}{6} - 1\right)\right] \simeq 0.630$$

$$B_3 = 3\left\{2[\zeta(3) - 1] + \left(\frac{\pi^2}{2} - 3\right) A_1 + 2I_4 - 2I_3(1 - B_1) - A_1^3\right\} \simeq 3.262$$

Here, $\zeta(n)$ is the Riemann zeta function, and $I_n = \int_0^\infty x^n K_0^n(x)\, dx$. The quantities I_3 and I_4 have been determined numerically[7]: $I_3 \simeq 0.586$ and $I_4 \simeq 1.052$. The expression for the dispersion follows from Eqs. (13) and (24):

$$\sigma^2(\tau) \simeq v_0^2 \frac{16\tau^2}{\ln^4 \beta\tau} \sum_{j=0}^{N} \frac{E_j}{\ln^j \beta\tau} \tag{25}$$

The first two coefficients of this asymptotic series are

$$E_0 = 4I_3 - (\pi^2/6 - 1) \simeq 1.699$$

$$E_1 = 4\{[\zeta(3) - 1] + (\pi^2/6 - 1)A_1 + 2I_4 - 2I_3(1 + B_1)\} \simeq 1.662$$

In the three-dimensional case, Eqs. (11) and (21) yield

$$\overline{v^2}(\tau) \simeq v_0^2 g\tau^2 \left[1 + \frac{4}{(\pi\tau)^{1/2}} + \frac{\ln \tau}{\tau} + O\left(\frac{1}{\tau}\right)\right] \tag{26}$$

$$\sigma^2(\tau) \simeq v_0^2 g\tau \ln \tau [1 + O(1/\ln \tau)] \tag{27}$$

For $d = 4$ the asymptotics for large time values, $\bar{v}^2(\tau)$, has the following form, according to Eq. (21):

$$\overline{v^2}(\tau) \simeq v_0^2 \cdot 64\tau^2 \left[1 + \frac{\ln \beta\tau}{2\tau} + O\left(\frac{1}{\tau}\right)\right] \tag{28}$$

The main terms of this equation and those of the relationship that follows from Eq. (15) for $\bar{v}(\tau)$ coincide. In this case the approximation used by us turns out to be too crude for calculating the dispersion of $\sigma^2(\tau)$, since the role of the configurations where the sinks are close to one another is no longer a small one. Thus, we find that the quantity of the order of $v_0\bar{v}(\tau)$ is an upper bound for the dispersion. Let us estimate a lower bound of the quantity under consideration. This quantity is served by the dispersion calculated on the assumption that the volume of a Wiener sausage is a random quantity with independent increments. The dispersion in this case is proportional to τ. A comparison of the two estimations indicates that at $d=4$

$$\sigma^2(\tau) \sim v_0^2\tau \tag{29}$$

In accordance with Eqs. (16) and (21), the same situation holds for spaces of greater dimensions.

Thus, the behavior of the main asymptotic term of $\sigma^2(\tau)$ at large time values in spaces of different dimensions follows the dependence

$$\sigma^2(\tau) = v_0^2\tau\phi_d(\tau)$$

where $\phi_d(\tau)$ is a constant value for $d=1$ and $d\geqslant 4$, and is proportional to $\tau/\ln^4\tau$ for $d=2$ and to $\ln\tau$ for $d=3$.

The time dependence of the relative fluctuation of the Wiener sausage volume $\tilde{\sigma}(\tau)=\sigma(\tau)/\bar{v}(\tau)$ makes it possible to judge the correlation of the increments in the volume visited by a Brownian particle at different moments. When they are independent, this quantity is proportional to $1/\tau^{1/2}$. This is exactly how $\tilde{\sigma}(\tau)$ behaves when $d\geqslant 4$. Starting from $d=3$, $\tilde{\sigma}(\tau)$ increases with a decrease in the dimension of space. It is proportional to $(\ln\tau/\tau)^{1/2}$ for $d=3$ and to $1/\ln\tau$ for $d=2$, and is independent of time for $d=1$. This circumstance reflects the significant role of self-intersections of a Wiener trajectory in a low-dimensional space.

4. ONE-DIMENSIONAL CASE

In the one-dimensional case the analog of the volume of a Wiener sausage corresponding to the trajectory of the center of a particle with dimension $2b$ is the amplitude of the particle's motion L

$$L \equiv v(W_t) = 2b + l_1(W_t) + l_2(W_t) \tag{30}$$

where $l_1(W_t)$ and $l_2(W_t)$ are the maximal distances from the starting point to the left and right, respectively, of the particle's center. Let us calculate the distribution of the probability density of the quantity L, $F_t(L)$. According to definitions in Eqs. (3) and (30), we have

$$F_t(L) = \langle \delta(L - 2b - [l_1(W_t) + l_2(W_t)]) \rangle$$
$$= \int_0^\infty \int_0^\infty \delta(L - 2b - (l_1 + l_2))\, \psi_t(l_1, l_2)\, dl_1\, dl_2 \tag{31}$$

Here, $\psi_t(l_1, l_2)\, dl_1\, dl_2$ is the portion of the trajectory for which the maximal deviations of the particle's center during time t to the left and right of the starting point are l_1 and l_2, respectively. The probability density $\psi_t(l_1, l_2)$ can be conveniently represented in the form

$$\psi_t(l_1, l_2) = \frac{\partial^2 P_t(l_1, l_2)}{\partial l_1\, \partial l_2} \tag{32}$$

where $P_t(l_1, l_2)$ is the probability that the particle starting from the origin of the coordinates will not leave the interval $(-(l_1 + b), (l_2 + b))$ during time t. This quantity is the integral over the interval $(-l_1, l_2)$ of the solution of a one-dimensional diffusion equation [Eq. (5)] which satisfies the initial condition in (6) and the boundary conditions $G(-l_1, t) = G(l_2, t) = 0$. This quantity is

$$P_t(l_1, l_2) = P_t(l = l_1 + l_2, \Delta l = l_1 - l_2)$$
$$= \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos[\frac{1}{2}\pi(2m-1)(\Delta l/l)]}{2m+1} \exp\left[-\frac{\pi^2 Dt}{l^2} (2m+1)^2 \right]$$

By using the definitions in Eqs. (31) and (32), it can be shown that

$$F_t(L + 2b) = \frac{\partial^2}{\partial L^2} [L P_t(L)]$$

where

$$P_t(L) = \frac{1}{2L} \int_{-L}^{L} P_t(L, \Delta l)\, d\Delta l$$
$$= \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \exp\left[-\frac{\pi^2 Dt}{L^2} (2m+1)^2 \right]$$

Since $P_t(L)$ depends on both the L and t arguments only through their dimensionless combination $y = L/\pi(Dt)^{1/2}$, it is convenient to express the probability density of the L amplitude through the probability density of the dimensionless quantity y

$$F_t(L + 2b) = \frac{1}{\pi(Dt)^{1/2}} f(y)$$

where

$$f(y)=\frac{16}{\pi^2 y^3}\sum_{m=0}^{\infty}\left[\frac{2}{y^2}(2m+1)^2-1\right]\exp\left[-\frac{(2m+1)^2}{y^2}\right] \tag{33}$$

This expression is convenient for analyzing the behavior of $f(y)$ at small values of y. The Poisson summation formula makes it possible to express $f(y)$ in the form of a series:

$$f(y)=4\pi^{1/2}\sum_{m=1}^{\infty}(-1)^{m-1}m^2\exp\left(-\frac{\pi^2 m^2 y^2}{4}\right) \tag{34}$$

which rapidly converges at large y. From Eqs. (33) and (34) it follows that

$$f(y)\simeq\begin{cases}\dfrac{32}{\pi^2}\dfrac{1}{y^5}\exp\left(-\dfrac{1}{y^2}\right) & \text{when} \quad y\ll 1\\[2ex] 4\pi^{1/2}\exp\left(-\dfrac{\pi^2}{4}y^2\right) & \text{when} \quad y\gg 1\end{cases}$$

Next we calculate the moments of the amplitude of a particle's Brownian motion. These quantities are linked to the moments of y through the relationship

$$M_j(t)=\int_0^{\infty}L^j F_t(L)\,dL=(2b)^j\sum_{k=0}^{j}c_j^k\left[\frac{1}{2}\left(\frac{Dt}{b^2}\right)^{1/2}\right]^k\tilde{M}_k \tag{35}$$

where $\tilde{M}_k=\int_0^\infty y^k f(y)\,dy$ is the kth moment of the quantity y, and c_j^k is the binominal coefficient. From Eq. (35) it follows that at times $t\gg b^2/D$ the moment of the jth order is proportional to $(Dt)^{j/2}$. By using Eqs. (33) and (34) we can determine all the moments of the y quantity. The results of the corresponding calculations are

$$\begin{gathered}\tilde{M}_0=1; \qquad \tilde{M}_1=\frac{4}{\pi^{3/2}}; \qquad M_2=\frac{8\ln 2}{\pi^2}\\ M_j=\frac{4}{\sqrt{\pi}}\left(\frac{2}{\pi}\right)^j\left(1-\frac{4}{2^j}\right)\Gamma\left(\frac{j+1}{2}\right)\zeta(j-1) \qquad \text{if} \quad j\geqslant 3\end{gathered} \tag{36}$$

The first relationship in (36) reflects the normalization per unit of the probability density $F_t(L)$ and $f(y)$. The second relationship leads to $M_1(t)=\bar{L}(t)$, a quantity which coincides with that calculated in Section 2 [Eq. (10)]. Relationships (35) and (36) make it possible to calculate the moments and cumulants of the quantity L of any order. In particular, for the dispersion we obtain

$$\sigma^2(t) = 8\left(\ln 2 - \frac{2}{\sqrt{\pi}}\right) Dt \simeq 0.452Dt \tag{37}$$

This precise expression coincides with the main term of the asymptotics for large time values [Eq. (23)] obtained in Section 3.

Thus, in the one-dimensional case it is possible to calculate completely the statistical characteristics of the Wiener sausage volume: the distribution function of this quantity, Eqs. (33) and (34), and the moments of any order, Eqs. (35) and (36).

5. CONCLUSION

The motion of the center of a Brownian particle corresponds to a random walk on the lattice, the number of steps n of which is large, $n \gg 1$.[8] During time t such a walk will accomplish

$$n = 2d(Dt/l^2) \tag{38}$$

steps on a d-dimensional simple cubic lattice with a period l. Here the walk will visit R_n different sites. It may be expected that R_n and the volume v visited by Brownian particles with a radius $b \gg l$ are related to one another. We shall discuss the question in more detail.

In the one-dimensional case the volume of a Wiener sausage according to Eq. (30) differs from the amplitude of the movement of the particle's center only by $2b$:

$$v = L = 2b + lR_n \tag{39}$$

It is known[7] that for $n > 1$

$$\bar{R}_n \simeq \left(\frac{8n}{\pi}\right)^{1/2}, \qquad \sigma_n^2 \simeq 4\left(\ln 2 - \frac{2}{\pi}\right) n$$

Taking account of these dependences and of the relationships (38) and (39), we get the same $\sigma^2(\tau)$ and $\bar{v}(\tau)$ as obtained in our calculations [cf. Eqs. (10), (37)].

In the d-dimensional case ($d \geqslant 2$) the situation is not so obvious. To analyze it, we shall compare the average values of the quantities under consideration. The asymptotics for large n values of the expectation value $\bar{R}_n$ is[7,9–11]

$$\bar{R}_n \simeq \begin{cases} \pi(n/\ln n) & \text{when} \quad d = 2 \\ \alpha_d n & \text{when} \quad d \geqslant 3 \end{cases} \tag{40}$$

where α_d is a constant which depends on the dimension of the space; $\alpha_3 \simeq 0.718$.[7] At times $t \gg b^2/D$ ($n \gg b^2/l^2$), when the volume visited by a particle considerably exceeds its own volume, $v \gg v_0$, Eqs. (38) and (40) make it possible to express the average volume of a Wiener sausage [Eqs. (11), (13), and (14)] in the form

$$\bar{v}(\tau) \simeq \gamma_d b^{d-2} \bar{R}_n l^2 \tag{41}$$

where γ_d is a numerical multiplier which depends on the dimension of the space; $\gamma_2 = 1$ and $\gamma_3 \simeq 2.917$.

In the two-dimensional case, as follows from Eq. (41), the volume visited by a particle is independent of its dimension and is determined only by the number of different sites visited by the particle's center. The visit by the center of each new site is accompanied by an increase in the volume of the Wiener sausage volume by Δv, which on the average equals the volume of the lattice cell l^2. Thus, the situation here is the same as in the one-dimensional case. This is explained by the fact that in low-dimensional space ($d = 1, 2$), due to the large number of self-intersections, the trajectory of the particle's center tightly fills the entire region visited by the particle.[4]

Important for an interpretation of the relationship (41) in spaces of higher dimensionality ($d \geqslant 3$) is the fact that the fractal dimension of a Wiener trajectory equals two.[2] This means that the characteristic trajectory of the particle's center tightly covers a certain two-dimensional surface, the average area of which, $R_n l^2 = 2d\alpha_d Dt$, depends on the dimension of the space. A typical Wiener sausage is a region formed as the result of the "growth" of this two-dimensional surface by a magnitude of the order of b in each of the $d - 2$ directions that are orthogonal to the surface. This explains the dependence of $\bar{v}(\tau)$ on the radius of the particle in Eq. (41).

If the quantities v and R_n were proportional to one another, as in the one-dimensional case [Eq. (39)], the ratio of their dispersion would be equal to the square of the ratio $\bar{v}(\tau)/\bar{R}_n$. The dependence σ_n^2 for $n \gg 1$ is[7,9–11]

$$\sigma_n^2 \simeq \begin{cases} 16.768 n^2/\ln^4 n & \text{when} \quad d = 2 \\ 0.215 n \ln n & \text{when} \quad d = 3 \\ \varepsilon_d n & \text{when} \quad d \geqslant 4 \end{cases} \tag{42}$$

where ε_d is a constant which depends on the dimension of the space. It can be seen that, according to Eqs. (25), (27), (29), (38), and (41), we have

$$\frac{\sigma^2(\tau)/\sigma_n^2}{[\bar{v}(\tau)/\bar{R}_n]^2} = \begin{cases} 1 & \text{when} \quad d = 2 \\ \eta_d (b/l)^2 & \text{when} \quad d \geqslant 3 \end{cases}$$

[4] It should be borne in mind that the probability of a return in the one-dimensional and the two-dimensional cases equals unity.[12]

where η_d is a constant ($\eta_3 = 14.387$). Thus, only in low-dimensional space ($d = 1, 2$) are the volume of a Wiener sausage and the number of different sites visited by a random walk proportional to one another.

APPENDIX

We shall calculate the death probability of a Brownian point particle during time $t \gg b^2/D$ in the presence of two sinks the configuration of which is determined by the conditions (19). This probability is defined as

$$u(\mathbf{r}_1, \mathbf{r}_2, t) = u(\mathbf{r}_1, t \mid \mathbf{r}_2) + u(\mathbf{r}_2, t \mid \mathbf{r}_1) \tag{A1}$$

where $u(\mathbf{r}_i, t \mid \mathbf{r}_k)$ is the probability of the particle's death on the sink located at $\mathbf{r}_i$ in the presence of one more sink located at $\mathbf{r}_k$ ($i, k = 1, 2$). The conditional probability $u(\mathbf{r}_i, t \mid \mathbf{r}_k)$ is always less than the unconditional probability $u(\mathbf{r}_i, t)$ introduced above [cf. Eq. (8)]. The latter is characterized by the death of the particle in a situation with one sink. The difference between them is the probability of the particle's death during time t in only one sink at $\mathbf{r}_i$ under the conditions of a preliminary visit by the particle to the b neighborhood of the $\mathbf{r}_k$ point.

Next we shall determine the approximate magnitude of this quantity. The probability that the particle will first visit the b neighborhood of the $\mathbf{r}_k$ point in the interval $(t', t' + dt')$ is $[\partial u(\mathbf{r}_k, t'' \mid \mathbf{r}_i)/\partial t']\, dt'$. The probability of its death after this toward the moment t is approximately $u(R, t - t')$, where $R = |\mathbf{r}_i - \mathbf{r}_k|$. The approximation consists in placing the particle visiting the b neighborhood of the $\mathbf{r}_k$ point at t' at the very point $\mathbf{r}_k$. For the magnitudes of the $\mathbf{r}_i$ and $\mathbf{r}_k$ considered here the error arising in this case is small. An integration with respect to all possible magnitudes of t' ($0 < t' < t$) determines the quantity sought. Thus, we obtain

$$u(r_i, t) - u(\mathbf{r}_i, t \mid \mathbf{r}_k) \simeq \int_0^t \frac{\partial u(\mathbf{r}_k, t' \mid \mathbf{r}_i)}{\partial t'} u(R, t - t')\, dt' \tag{A2}$$

By using the Laplace transform, we obtain from Eq. (A2) a system of linear equations for the unknown quantities $u(\mathbf{r}_i, s \mid \mathbf{r}_k)$:

$$u(\mathbf{r}_i, s \mid \mathbf{r}_k) + su(\mathbf{r}_k, s \mid \mathbf{r}_i)\, u(R, s) = u(r_i, s)$$

Its solution has the following form:

$$u(\mathbf{r}_i, s \mid \mathbf{r}_k) = \frac{u(r_i, s) - su(R, s)\, u(r_k, s)}{1 - s^2 u^2(R, s)} \tag{A3}$$

From Eqs. (A1) and (A3) we obtain Eq. (20) (see Section 3).

NOTE ADDED IN PROOF

Thanks to the referee we found out about the recently published paper: J.-F. Le Gall, Ann. of Probability **16**:991 (1988), as well as about the papers: F. Spitzer, Z. Wahrscheinlichkeitstheorie und verwandte Gebiete **3**:110 (1964) and R. K. Getoor, Z. Wahrscheinlichkeitstheorie und verwandte Gebiete **4**:248 (1965). These papers are devoted to the same problem but in a more general form. The solutions presented in these papers fully agree with the results obtained by us. Lower generality of the problem statement enabled us to carry out a more detailed study on some of the values considered. In particular, the mean value of the Wiener sausage volume at the arbitrary time instant was found but not only the asymptotics of this value with $t \to \infty$ calculated in the above mentioned papers. Besides, our method of calculation of the moments of the Wiener sausage volume is different from the methods used in the papers of Spitzer, Getoor and Le Gall.

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